Quadratics

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Class Discussion

$x^2 - 5x + 6$, $x^2 - 12x - 540$ — use divisibility. $3t^2 + 11t + 10 = 0$. Find the ratio x/y if $x^2 - 5xy + 6y^2 = 0$.

Warm-Up

Exercise 1. There are three people (Alex, Brook and Cody), one of whom is a knight, one a knave, and one a spy. The knight always tells the truth, the knave always lies, and the spy can either lie or tell the truth. Alex says: "Cody is a knave." Brook says: "Alex is a knight." Cody says: "I am the spy."

Who is the knight, who the knave, and who the spy?

Exercise 2. You need to take socks from a drawer in a very dark room. You are told that the drawer contains 6 blue socks, 5 red socks, and 10 white socks. What is the smallest number of socks you need to take before you can be sure to have at least one matching pair?

Quadratics

Exercise 3. AHSME. How many integers x satisfy the equation: $(x^2 - x - 1)^{x+2} = 1$?

Exercise 4. Consider the set of parabolas defined by $y = x^2 + px + q$, where $p + q = 2011$. Prove that all those parabolas pass through the same point on the plane.

Exercise 5. AHSME. The sum of the squares of the roots of the equation $x^2 + 2hx = 3$ is 10. Find $|h|$.

Exercise 6. The quadratic $y = ax^2 + bx + c$ doesn't have real roots and $a + b + c > 0$. Is c positive or negative?

Exercise 7. AMC 10. Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + 1/b$ and $b + 1/a$ are the roots of the equation $x^2 - px + q = 0$. What is q?

Exercise 8. Both roots of the equation $x^2 + px + q = 0$ are integers. Find p and q, given that p and q are prime numbers.

Exercise 9. For which a, one of the roots of the equation $x^2 - \frac{15}{4}x + a^3$ is a square of the other root?

Exercise 10. HMMT. Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.

Competition Practice

Exercise 11. 2006 USAMO. For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k + 1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $N/2$.

Challenge Problems

Exercise 12. Six segments are such that you can make a triangle out of any three of them. Is it true that you can build a tetrahedron out of all six of them?

Exercise 13. An ant is sitting on the corner of a brick. A brick means a solid rectangular parallelepiped. The ant has a math degree and knows the shortest way to crawl to any point on the surface of the brick. Is it true that the farthest point from the ant is the opposite corner?

Pi Day

Happy Pi Day!

Jokes

A joke:

> —What do you get when you take the sun and divide its circumference by its diameter?
> —Pi in the sky.

A limerick:

> 'Tis a favorite project of mine
> A new value of pi to assign.
> I would fix it at 3
> For it's simpler, you see,
> Than 3 point 1 4 1 5 9.

A joke:
Said the Mathematician, "Pi r squared."
Said the Baker, "No! Pie are round, cakes are square!"

Mnemonics

- How I wish I could calculate pi.
- How I wish I could enumerate Pi easily, since all these horrible mnemonics prevent recalling any of pi's sequence more simply.
- How I want a drink, alcoholic of course, after the heavy chapters involving quantum mechanics. One is, yes, adequate even enough to induce some fun and pleasure for an instant, miserably brief.

Do you see a problem with mnemonics?
The problem is with digit zero. Luckily the first zero occurs at 33 digit.
Do we need many digits of pi?

Approximating Pi

When we try to approximate a number as a fraction m/n with a given n, what is our precision? Answer: $1/2n$. Continued fractions allow us to be much more precise: m/n approximation by continued fraction has precision $1/n^2$.

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \ddots}}}}}}}}}}}.$$

The consecutive convergents are: 3/1, 22/7, 333/106, 355/113, and so on.

Buffon's needle: Drop a needle of length 1 repeatedly on a surface containing parallel lines drawn 1 units apart. If the needle is dropped n times and x of those times it comes to rest crossing a line, then one may approximate π using: $2n/x$.

Randomness of Pi

So far the digits of Pi passed all statistical randomness tests. What does it tell us about the statistics?

Property of randomness—can't compress. Digits of Pi highly non-random: we can compress to one Greek letter.

But it is not even proved that all the digits occur infinitely many times.

Formulae

An integral:

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}.$$

Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Riemann zeta function:

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

Leibniz formula for pi:

$$\sum_{n=0}^{\infty} \left(\frac{(-1)^n}{2n+1}\right)^1 = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \arctan 1 = \frac{\pi}{4}$$

Vieta's formula:

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \cdots = \frac{2}{\pi}.$$

Stirling's approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Euler's identity:

$$e^{i\pi} + 1 = 0.$$

Efficient infinite series:

$$\sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!} = \frac{\pi}{2}.$$