Symmetric Polynomials

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"I did part of my homework; the part I have left to do, is 0.999999999..."

Class Discussion

A polynomial $P(x, y)$ is called symmetric if it is equal to $P(y, x)$. Any symmetric polynomial in two variables x and y can be expressed as a polynomial of $x + y$ and xy.

A polynomial $P(x, y, z)$ is called symmetric if $P(x, y, z) = P(x, z, y) = P(y, x, z) = P(y, z, x) = P(z, x, y) = P(z, y, x)$. A symmetric polynomial in three variables can be expressed as a polynomial of: $x + y + z$, $xy + yz + xz$, and xyz.

Roots and coefficients. If a, b, c are roots of the polynomial $x^3 + px^2 + qx + r = 0$, then $x^3 + px^2 + qx + r = (x - a)(x - b)(x - c)$ and $p = -a - b - c$, $q = ab + ac + bc$, and $r = -abc$.

Warm Up

Exercise 1. 2009 Moscow Olympiad 6th Grade. Genetically engineered octopuses with an even number of arms always tell the truth, the ones with an odd number of arms always lie. Four of them had a chat. "I have 8 arms," the green octopus bragged to the blue one. "You have only 6!" — "It is I who has 8 arms," countered the blue octopus. "You have only 7!" — "The blue one really has 8 arms," the red octopus said, confirming the blue one's claim. He went on to boast, "I have 9 arms!" — "None of you have 8 arms," interjected the striped octopus. "Only I have 8 arms!"

Who has exactly 8 arms?

Exercise 2. 2009 Moscow Olympiad 7th Grade. Genetically engineered octopuses with an even number of arms always tell the truth, the ones with an odd number of arms always lie. Only the ones with six, seven or eight arms are allowed to serve Neptune. Once four octopuses who were working as guards at Neptune's palace started a conversation: The blue one said, "All together we have 28 arms." The green one said, "All together we have 27 arms." The yellow one said, "All together we have 26 arms." The red one said, "All together we have 25 arms."

How many arms does each of them have?

Exercise 3. 2009 Moscow Olympiad 8th Grade. A teacher wrote a self-referencing statement on the blackboard:

> In this sentence ...% of the digits are divisible by 2, ...% are divisible by 3, and ...% of the digits are divisible by 2 and 3.

Fill in the blanks, so that the statement is true.

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Exercise 4. In the equation $3x^2-3x+1=0$, what is the sum of the squares of the roots?

Exercise 5. 1991 AHSME. If the sum of two numbers is 1 and their product is 1, then what is the sum of their cubes?

Exercise 6. If a and b are the roots of $x^2+px+q=0$, then find each of the following in terms of p and q: a^2+b^2, a^3+b^3, a^2b+ab^2, $a-b$.

Exercise 7. Find x^4+1/x^4 if $x-1/x=5$.

Exercise 8. 2007 HMMT. Two reals x and y are such that $x-y=4$ and $x^3-y^3=28$. Compute xy.

Exercise 9. 2006 HMMT. Let a, b, c be the roots of $x^3-9x^2+11x-1=0$, and let $s=\sqrt{a}+\sqrt{b}+\sqrt{c}$. Find s^4-18s^2-8s.

Exercise 10. 2008 HMMT. Let $f(x)=x^3+x+1$. Suppose g is a cubic polynomial such that $g(0)=-1$, and the roots of g are the squares of the roots of f. Find $g(9)$.

Exercise 11. Find all possible values of x^3+1/x^3 given that $x^2+1/x^2=7$.