Remainders

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It was mentioned on CNN that the new prime number discovered recently is four times bigger then the previous record.

Class Discussion

Remainders. Remainders of powers. Remainders of squares. Prove that $6^{2n+1} + 1$ is divisible by 7.

Warm Up

Exercise 1. My dog Fudge likes books. In the morning he brought two books to his corner and three more books in the evening. How many books will he read tonight?

Exercise 2. The population of the island of Pianosa is 100. Some of the inhabitants always lie, the others always tell the truth. Each islander worships one of three gods: the Sun god, the Moon god, or the Earth god. One day a visiting anthropologist asked each inhabitant the following questions:

1. Do you worship the Sun god?
2. Do you worship the Moon god?
3. Do you worship the Earth god?

There were 60 "yes" answers to the first question, 40 "yes" answers to the second question, and 30 "yes" answers to the third. How many liars live on the island?

Remainders

Exercise 3. Prove that the remainder of a prime number modulo 30 is either prime or 1.

Exercise 4. Find the remainders of 2^{2014} modulo 3, 5, 7, and 9.

Exercise 5. Let m and n be integers. Prove that $mn(m+n)$ is even.

Exercise 6. Find all prime numbers p and q such that $p^2 - 2q^2 = 1$.

Exercise 7. Prove that there are 1000 consecutive composite numbers.

Exercise 8. Prove that 3, 5 and 7 is the only triple of twin primes.

Exercise 9. Prove that for $n > 2$, numbers $2^n - 1$ and $2^n + 1$ can't be both prime.

Competition practice

Exercise 10. 2005 AMC 10A. Problem 16. The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property?

Exercise 11. 2005 AMC 10A. Problem 21. For how many positive integers n does $1 + 2 + \cdots + n$ evenly divide $6n$?

Exercise 12. 2006 AMC 10B. Problem 11. What is the tens digit in the sum $7! + 8! + 9! + \cdots + 2006!$?

Exercise 13. HMMT. Find the largest square of the form $1! + 2! + 3! + \ldots + n!$.

Challenge Problems

Exercise 14. Give an example of 5 numbers such that their product is not 0, and such that if you subtract 1 from each number, then their product will remain the same.