Test Solutions

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Exercise 1. 1 point. The professor is watching across a field how the son of the professor's father is fighting with the father of the professor's son. How is this possible?

Answer: The professor is a female. She watches her brother fighting with her husband.

Exercise 2. 1 point. Bob spent the first Tuesday of the month hiking in Himalayas. He spent Tuesday after the first Monday of the same month at a conference in Seattle. In the next month he spent the first Tuesday on cruise in Black sea. He spent Tuesday after the first Monday of that month celebrating his birthday in Boston. When is his birthday?

Answer: The first Tuesday of the month and Tuesday after the first Monday of the month are different only if Tuesday is the first day of the month. Tuesday can be the first day of the month twice in a row only if the first month is February. His birthday is on March 8.

Exercise 3. 1 point. My friend Alice will celebrate her x birthday in the year x^2. When was she born?

Answer: The next square year is $2025 = 45^2$. Alice was born in 1980.

Exercise 4. 1 point. Two two-digit prime numbers are reverses of each other. Their difference is a square. What are the numbers?

Answer: These prime numbers have to end and start with 1, 3, 7, or 9. The two numbers could be equal, then they are 11. In any case their difference has to be divisible by 9. It is also even. Hence, it is divisible by 36. The numbers are 37 and 73.

Exercise 5. 1 point. What is the smallest positive integer that is not a factor of 100!?

Answer: It is the smallest prime number over 100: 101.

Exercise 6. 1 point. Multiplications are expensive. You need to program a function to calculate x^{33} when x is given. What is the smallest number of multiplications that you need?

Answer: In five multiplications we can get the squares: x^2, x^4, x^8, x^{16}, and x^{32}. We need one more multiplication: x times x^{32}. The answer is 6.

Exercise 7. 1 point. What is the maximum number of edges that a planar graph with 44 vertices can have?

Answer: We actually saw this sequence before: Every new vertex add three more edges. If you forgot, then to maximize the number of edges, every face has to be a triangle. For 3 vertices we have 3 edges. We can add a new vertex inside one of the existing triangles and connect to the vertices of the triangle. This way each new vertex adds three edges. The answer is $(44-2)\dot{3} = 126$.

Exercise 8. 1 point. I can predict the score of every basketball game before it starts. How?

Answer: The score of any game before it starts is 0:0.

Exercise 9. 2 point. Someone put 30 dots inside a square and connected the dots with each other and with the vertices of the square in such a way that the square became divided into triangles. How many triangles are there?

Answer: We had a similar problem before. If you remember that the number doesn't depend on the configuration of points then you can calculate it using an example. One point inside the square generates 4 triangles. Each new point adds two more triangles. The answer is 62.

Or we can use the Euler's formula. Suppose there are x triangles. There are total of 34 vertices, the number of faces is $x+1$: the triangles and the outer face. The number of edges is $(3x+4)/2$. Each triangle has 3 edges, the outer face has 4 edges and each is counted twice. The Euler formula gives: $34 + x + 1 = 2 + (3x + 4)/2$. Hence, $x = 62$.

Exercise 10. 2 points. There are four silver coins marked 1, 2, 3, and 5. They are supposed to weigh the number of grams that is written on them. One of the coins is fake and is lighter than it should be. Find the fake coin using the balance scale twice. Explain.

Answer: First we compare 2 and 3 with 5. If the scale balances, the fake coin is 1. If 5 is lighter it is fake. If 2 and 3 are lighter then compare 1 and 2 with 3 to distinguish between them.

Exercise 11. 2 points. Can a power of 2 have the same number of zeros, ones, twos, ..., nines? Explain.

Answer: Such a power of two would have the sum of the digits that is a multiple of 45. Hence, it will be divisible by 9.

Exercise 12. 2 points. In the following sentence replace the dots with a digit, so that the statement is true. "All the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 participate in this sentence, in particular, digit 0 — ... times, digit 1 — ... times, digit 2 — ... times, digit 3 — ... times, digit 4 — ... times, digit 5 — ... times, digit 6 — ... times, digit 7 — ... times, digit 8 — ... times, and digit 9 — ... times.

Answer: Each digit already participates at least two times. Hence, each number in question is at least two. Hence, the numbers of zeros and ones is 2, and the number of twos is at least 4.

In addition, the sum of the numbers we need to put in is 30. So the sum of the digits that we place for 3 through 9 is not more than 22. If two appears 4 times, then others are not twos. They have to be six threes and one four. Then three appears 8 times, a contradiction. Two has to appear at least 5 times. If two appears 5 times, then the leftover numbers need to be distributed as 1 two, 5 threes and 1 four, a contradiction. If two appears 6 times, then we get 2 more twos, 4 threes and 1 four, a contradiction. Continue. If two appears 7 times, we get 3 more twos, 3 threes and 1 four, a contradiction. If two appears 8 times, we get 4 more twos, 2 threes, and 1 four. This works. The numbers are 2, 2, 8, 4, 3, 2, 2, 2, 3, 2.

Exercise 13. 3 points. Baron Munchausen has a habit of hunting wild ducks every day. For several days in a row he truthfully told to his cook "Today I caught more ducks than two days ago and fewer than a week ago." Given that he never lies, for how many maximum days in a row can he repeat that?

Answer: He can say this truthfully for 6 days. We label the number of ducks he caught on the ith day as d_i. Suppose that he starts saying this sentence on day 3. If we have

$$d_1 = 10, d_2 = 0, d_3 = 11, d_4 = 1, d_5 = 12, d_6 = 2, d_7 = 13, d_8 = 3,$$

then he told the truth for 6 days in a row. Now suppose that he said this statement truthfully for 7 days. Then, we must have

$$d_1 < d_3 < d_5 < d_7 < d_9 < d_2 < d_4 < d_6 < d_8 < d_1.$$

This is a contradiction.

Exercise 14. 3 points. There are 12 people in the room. Some of them are liars and some truth-tellers. The first person said, "There are no truth-tellers here.". The second person said, "There are no more than 1 truth-teller here." The third person said, "There are no more than 2 truth-tellers here." And so on. The 12-th person said, "There are no more than 11 truth-tellers here." How many truth-tellers are in the room? Explain.

Answer: If person k tells the truth, then person $k+1$ tells the truth. Let n be the number of the last liar. Then there $12 - n$ truth-tellers. And there are no more than $12 - n$ truth-tellers. So the person with number $12 - n + 1$ is the first truth-teller. That means, $12 - n + 1 = n + 1$, and there are 6 truth-tellers in the room.

Exercise 15. 2 points. Shauna was killed one Sunday morning. The police questions everyone about their activities during the time of death. Here are the replies:

- Alyssa was doing laundry
- April was getting the mail
- Mark was planting in the garden
- Reggie was cooking

Who killed Shauna?

Answer: April lied: there is no mail on Sunday.