Mathpath 2006 Qualifying Quiz

Instructions

Do as many of the 8 problems below as you can. If you can do 4 problems (in whole or in sum of parts) you should definitely apply! Keep thinking about each problem until the approach to the solution comes to you, sometimes after several days. You may ask others to help you understand the statements of the problems, but the actual solutions must be your own. Do not be disappointed if you spend a long time on some problem but never solve it. This happens to famous mathematicians too!

Use 8.5×11 paper (ruled or unruled) and write on one side only. Please start each problem on a new sheet. The sheets and the problems should be numbered. You need not copy the statements of problems. However, your solutions should show all the steps in your reasoning and in your computations. The steps are more important than the answer. Correct answers without supporting reasoning will receive no credit.

Also, additional generalizations and additional solutions are welcome and may count significantly. That is, if you see one of these problems as a special case of another problem, and can solve the bigger problem, do so. Or, if after solving the given problem completely, you see a completely different solution, submit it.

Communication is an important part of MathPath. Imagine you are writing to a friend who knows about as much mathematics as you but who has not thought about these problems before. You want your friend to understand your solutions as easily as possible, so your work must be clear. In particular, long solutions with lots of cases are hard to follow. Shorter, more direct solutions are preferred (but not if they are shorter simply by leaving out reasons). So, if your first solution to a problem is long and complicated, see if you find a short direct solution, and submit only that. Mathematicians say that such short direct solutions are *elegant*.

For more discussion and examples of good and not-so-good solutions from earlier MathPath quizzes, click HERE.

Click HERE for any posted hints and clarifications.

1 What is the largest integer that divides $p^4 - 1$ for all primes $p > 3$? Justify your answer.

2 Alice and Bob alternate tossing a fair coin, Alice going first. The first one of them to toss a Head immediately after the other tosses a Tail is the winner. Find the probability that

(a) Alice wins on her first toss (we're checking that you read carefully).

(b) Bob wins on his first toss.

(c) Alice wins on her third toss.

(d) Bob wins on his 5th toss.

(e) Alice wins.

3 Suppose that a, b, c, d are real numbers such that

$$\begin{aligned} a^2 + b^2 &= 1 \\ c^2 + d^2 &= 1 \\ ac + bd &= 0. \end{aligned}$$

Show that

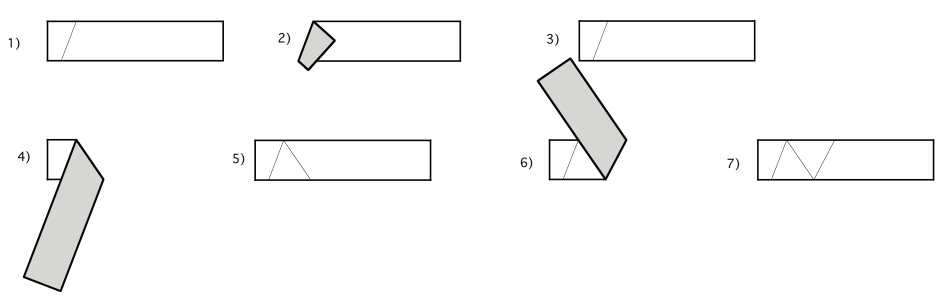
$$\begin{aligned} a^2 + c^2 &= 1 \\ b^2 + d^2 &= 1 \\ ab + cd &= 0. \end{aligned}$$

This problem can get very messy, but doesn't have to. Strive to find an elegant and complete solution.

4 Consider the following "recipe" for folding paper to get equilateral triangles (see figure below):

1. Start with a long strip of paper, and visualize a crease for folding (thin line). The crease can have any angle.
2. Hold the top left corner and fold it down on this crease so that the corner is now below the bottom of the strip.
3. Unfold. You now actually have a crease (shown with thin line).
4. Now grab the right end of the strip and fold DOWN so that the top side of the strip is along this crease.
5. Unfold. You now have two creases.
6. Now grasp the right end and fold UP so that the bottom side of the strip is along the most recently created crease.
7. Unfold. You now have three creases.
8. Repeat steps 4–7. Your creases will now be equilateral triangles!

Comment on this procedure. Does it work? Does it almost work? Explain!



5 Let A, B, C, D be the vertices of a regular tetrahedron of side length 1. (A regular tetrahedron is a polyhedron with four congruent equilateral triangular faces. At each of its four vertices, three faces meet.) Let P be the midpoint of side AB. Let R be the midpoint of side CD

(a) Find the length of PR.

(b) Find the angle that PR makes with the plane of face ABC.

6 Let $Z(n)$ denote the number of zeros at the end of $n!$. For example, $Z(11) = 2$, since $11! = 39916800$ ends with 2 zeros.

(a) Compute $Z(2006)$, without a calculator or computer.

(b) We call a positive integer t a Z-fac number if there is an integer m such that $Z(m) = t$. Prove that, for all n, the number of integers from 1 to $Z(n)$ that are NOT Z-fac numbers is equal to $Z(\lfloor n/5 \rfloor)$. (The notation $\lfloor x \rfloor$ denotes the *floor* of x, i.e., the greatest integer less than or equal to x. For example, $\lfloor \pi \rfloor = 3$ and $\lfloor 17 \rfloor = 17$.)

7 In Klopstockia, bees live in hives, obeying the following three rules:

1. Every pair of bees has exactly one hive in common.
2. Every pair of hives has exactly one bee in common.
3. There are at least 4 bees such that no 3 are in the same hive.

Answer the following questions.

(a) Find the smallest possible number of bees in Klopstockia. Justify your answer.

(b) Find the smallest possible number of hives in Klopstockia. Justify your answer.

(c) See if you can discover anything else about bees and hives in Klopstockia. Explain and justify what you find.

8 You are given 5 dots arranged on a circle, and told to draw segments between pairs of the points to connect all the dots. It is always possible to do this with 4 segments. However, suppose you are required to use 5 segments, that is, the dots should not all be connected until you draw your fifth segment. For instance, if the dots are numbered 1,2,3,4,5, one way to do this is to draw the following sequence of segments: 12, 34, 24, 13, 35. Another sequence would be 34, 13, 12, 24, 35; it uses the same segments but in a different order. (But careful: some other orders of these 5 segments don't count; why not?). Another sequence, using some different edges, is 23, 24, 25, 34, 15. Note that you may not draw the same segment twice. In other words, 12, 23, 23, 34, 45 uses only 4 segments, not 5.

How many sequences are there which take 5 segments to connect all 5 dots?