Invariants 3

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Class Discussion

Romeo and Juliet.

Most of todays problems are translated from http://problems.ru/.

Warm Up

Exercise 1. There are six numbers on the blackboard: 1, 2, 3, 4, 5, 6. In one move you are allowed to add 1 to any two of them. How many moves do you need to make all numbers equal each other?

Exercise 2. A frog jumps along the line. First it jumped 1 cm, then 3 cm in the same or the opposite direction, then 5 cm. It continues with the sequence of odd numbers. Can it be back at the beginning after 57 jumps?

Exercise 3. A knight starts on the chessboard at the $a1$ cell. Can it tour all the cells of the chessboard exactly once and end at the $h8$ cell?

Exercise 4. Numbers are put into an m by n table so that the sum in every row is 1, as well as the sum in every column is 1. Prove that $m = n$.

Exercise 5. Alice and Bob are playing a game. The game starts with two piles of pebbles: with 10 pebbles and 15 pebbles. Each turn a person is allowed to divide one pile into two smaller piles. Alice starts, and the person who doesn't have a move loses. Can Bob win?

Exercise 6. I've put 100 coins in one row. In one move you are allowed to switch two coins that are separated by exactly one coin in between them. How can you put all the coins in the reverse order?

Competition Practice

Exercise 7. 1998 Russian Olympiad. Alice has a standard deck of 52 cards face down. Alice guesses a suit, then opens the top card. If she guesses correctly, she gets a point. Alice continues through the whole deck. Alice devised her strategy. She names the suit such that the suit has more or equal cards left than any other suit. Prove that Alice will get 13 points by the end of the deck.

Exercise 8. 1988 Tournament of Towns. Every vertex of a cube is assigned a number $+1$ or -1. Every face has a number that is the product of all the numbers in its corners. Then the 14 numbers are summed up (all the vertices and faces). Can the sum be 0?

Challenge Problems

Exercise 9. Numbers 1 and 2 are written on the board in the office of the math professor Bob Enstein. Every day Bob's graduate student has to replace two numbers on the board with their arithmetic and harmonic means. If a and b are numbers, then $(a+b)/2$ is their arithmetic mean, and $2/(1/a+1/b)$ is their harmonic mean. Once Bob came to his office and noticed that one of the numbers got erased. What is the erased number if the other number is 941664/665857? Is it possible that Bob will ever see 35/24 as one of the numbers on the board?

Exercise 10. An ace of spades is on the table face up. You are allowed to roll it over the edge several times. Can it end up in the initial spot face down? Can it end up in the initial spot face up, but with the picture upside down?

Exercise 11. Can you cover a 10 by 10 board with one corner cut out by 1 by 3 rectangles? What about an 8 by 8 board with one corner cut out?

Exercise 12. Neverland has n towns so that all pairwise distances between them are different. Alice likes traveling. She starts from her home town A, then she moves to town B that is the farthest away from A. Then she moves to town C that is the farthest away from B. If C is not the same as A, prove that she will never be back home.