Tetramodules over the Hopf algebra of regular functions on a torus.

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Introduction.

The definition of a tetramodule appeared in my joint work with Joseph Bernstein [B-KH]. In this paper we initiated an axiomatic approach to the construction of the quantum group $SL_q(2)$. We hope to use this approach more universally.

The notion of a tetramodule seems to be interesting in itself. The goal of this paper is to describe the basic properties of tetramodules.

I am thankful to Joseph Bernstein who encouraged me to write this paper. I would like to thank Victor Kac and David Kazhdan for helpful discussions.

The author was supported by an NSF Grant # DMS-9306018.

1. Motivations.

1.1. Let S be a Hopf algebra. We would like to study pairs (A, I), where A is a Hopf algebra (with multiplication m and comultiplication Δ) and $I \subset A$ a two-sided Hopf ideal, such that A/I is isomorphic to S as a Hopf algebra.

1.2. Note that the comultiplication $\Delta : A \to A \otimes A$ leads to two S-comodule structures on A:

$$\begin{aligned} c_\ell &: A \to S \otimes A \qquad & c_\ell &= (pr \otimes id)\Delta \\ c_r &: A \to A \otimes S \qquad & c_r &= (id \otimes pr)\Delta \end{aligned}$$

where pr is the natural projection $A \to S = A/I$.

1.3. Consider the associated graded algebra $gr\, A$:

$$gr\, A = \bigoplus_{0 \le n} gr_n\, A,$$

where

$$gr_n\, A = I^n / I^{n+1} \ .$$

It is easy to see that $gr\, A$ inherits the structure of a graded Hopf algebra with $gr_0 A$ equal to S. In particular, grA has the structure of a graded S-bicomodule.

1.4. We have two natural S-module structures on grA. These structures commute and preserve $gr_n A$.

The S-bicomodule and S-bimodule structures are compatible: for any $s \in S,\ x \in grA$

$$c_\ell(xs) = c_\ell(x)\Delta s \ .$$

The other three relations are of the same type:

$$c_\ell(sx) = \Delta s \cdot c_\ell(x)$$

$$c_r(xs) = c_r(x)\Delta s$$

$$c_r(sx) = \Delta s \cdot c_r(x).$$

1.5. *Definition.* We call the linear space V an *S-tetramodule* if V is equipped with commuting left and right S-module structures, commuting left and right S-comodule structures, and the S-bimodule and S-bicomodule structures are compatible (see 1.4).

1.6. *Example.* Tetramodules first appeared in my work with Joseph Bernstein [B-KH]; where A was the Hopf algebra of regular functions on the quantum group $SL_q(2)$, and S was the Hopf algebra of regular functions on the one-dimensional torus H. In this case an S-comodule structure defines an algebraic representation of H.

2. Definition of tetramodule.

2.1. Let us rewrite the definition of S-tetramodule.

Definition. Given a Hopf algebra S, an S-tetramodule is a vector space V equipped with four morphisms

$$\begin{array}{lclcl} m_\ell & : & S\otimes V & \to & V \\ m_r & : & V\otimes S & \to & V \\ c_\ell & : & V & \to & S\otimes V \\ c_r & : & V & \to & V\otimes S \end{array}$$

satisfying the following relations – $H1, H2, H3$:

$H1$. The morphism m_ℓ (resp. m_r) defines the structure of a left (resp. right) S-module on V. This means that it is associative, and the element $1 \in S$ acts as the identity.

$H1'$. The actions m_ℓ and m_r commute on V. This means that the operator $m_r(m_\ell \otimes id)$ equals the operator $m_\ell(id \otimes m_r)$ as an operator from $S \otimes V \otimes S$ to V.

$H2$. The morphism c_ℓ (resp. c_r) defines the structure of left (resp. right) S-comodule on V. This means that it is coassociative, and the counit acts as the identity.

$H2'$. The actions c_ℓ and c_r cocommute on V. This means that the operator $(id \otimes c_r)c_\ell$ from V to $S \otimes V \otimes S$ equals the operator $(c_\ell \otimes id)c_r$.

$H3$. The connection between the S-module and the S-comodule structures:

$H3rl$. This axiom describes the compatibility of m_r and c_ℓ: The morphism $m_r : V \otimes S \to V$ is a morphism of left S-comodules. In other words the following diagram commutes:

$$\begin{tikzcd}
& V \arrow[dr, "c_\ell"] & \\
V \otimes S \arrow[ur, "m_r"] \arrow[d, "c_\ell \otimes \Delta"] & & S \otimes V \\
(S \otimes V) \otimes (S \otimes S) \arrow[rr, "S_{2,3}"] & & (S \otimes S) \otimes (V \otimes S) \arrow[u, "m \otimes m_r"]
\end{tikzcd}$$

Note that this diagram is also equivalent to the requirement that the morphism $c_\ell : V \to S \otimes V$ is a morphism of right S-modules.

Similarly, we define the connection axioms $H3ll, H3lr, H3rr$ describing the compatibility of pairs (c_ℓ, m_ℓ), (c_r, m_ℓ), (c_r, m_r).

3. Decomposition of tetramodule.

3.1. From now on we consider only the case when S is the Hopf algebra of regular functions on a torus H. In this case we can give an explicit description of the category of S-tetramodules [B-KH].

3.2. We use the following standard

Lemma. Let W be an S-module equipped with the compatible algebraic action of the group H. Then $W = S \otimes W^H$, where W^H is the space of H-invariants.

3.3. Let us apply this lemma to our case. Let V be an S-tetramodule. Applying lemma 3.2 to the right action of H on V and the right multiplication by S we can write V as $V = V^H \otimes S$.

3.4. Now let V be an S-tetramodule $V = V^H \otimes S$. We want to describe an S-tetramodule structure on V in terms of some structures on V^H.

The right action of H on V^H is trivial. It is clear that V^H is ad_H-invariant, so the left action of H on V^H coincides with the ad_H action. Hence, knowing the ad_H action on V^H, we can reconstruct the left and right actions of H on V.

The right action of S on V is defined by decomposition $V = V^H \otimes S$. Now we have to reconstruct the left action of S on V.

Let Λ be the lattice of characters of H. Then $\Lambda \subset S$ is a basis of S. For $\lambda \in \Lambda$ consider operators $m_\ell(\lambda)$ and $m_r(\lambda)$ of left and right multiplications by λ in V, and set $L(\lambda) = m_\ell(\lambda) m_r(\lambda)^{-1}$. Then operators $L(\lambda)$ commute with the right and the left action of H and hence preserve the subspace V^H.

So we have defined a homomorphism L of Λ into automorphisms of V^H, commuting with ad_H. Knowing L we can reconstruct the left action of S on V.

3.5. *Summary.* Let S be the Hopf algebra of regular functions on a torus H. Then the functor $V \to V^H$ gives an equivalence of the category of S-tetramodules with the category of algebraic H-modules equipped with the commuting action L of the lattice Λ .

4. Tetramodules over $S \otimes S$.

4.1. Let B be a Hopf algebra. We call an imbedding $i : B \to S$ a Hopf imbedding if it is closed under multiplication: $i \cdot m_B = m_S \cdot (i \otimes i)$ and respects comultiplication: $(i \otimes i) \cdot \Delta_B = \Delta_S \cdot i$.

We call a projection $p : S \to B$ a Hopf projection if it is closed under comultiplication: $(p \otimes p) \cdot \Delta_S = \Delta_B \cdot p$ and respects multiplication: $p \cdot m_S = m_B \cdot (p \otimes p)$.

4.2. Let V be an S-tetramodule. A Hopf imbedding $i : B \to S$ gives us a B-bimodule structure on V. A Hopf projection $p : S \to B$ gives us a B-bicomodule structure on V.

Theorem. If $p \cdot i = \mathrm{id}$, then these bimodule and bicomodule structures are compatible.

Proof. Let us prove, for instance, the compatibility of right B-module and right B-comodule structures on V. The right B-comodule structure on V is defined as the composite map:

$$V \xrightarrow{c_r} V \otimes S \xrightarrow{id \otimes p} V \otimes B.$$

We have to prove that this composition defines a homomorphism of B-modules. The first map defines a homomorphism of S-modules, and, hence, of B-modules.

The B-module structure on $V \otimes B$ is defined as $(i \otimes id) \cdot \Delta_B$. The B-module structure on $V \otimes S$ is defined as $\Delta_S \cdot i$, which is equal to $(i \otimes i) \cdot \Delta_B$ for Hopf imbedding i. So, all is left to prove is that $p : S \to B$ defines a homomorphism of B-modules. It is by definition of p that p defines a homomorphism of S-modules. The induced B-module structure on B is equal to $p \cdot i(B)$, which is equal to the existing B-module structure on B.

4.3. Denote by η the unit in S: $\eta : k \to S$, and by ε the counit in S: $\varepsilon : S \to k$. There are three natural Hopf imbeddings $S \to S \otimes S$:

(i) left - $i_\ell : S = S \otimes k \xrightarrow{id \otimes \eta} S \otimes S$;

(ii) right - $i_r : S = k \otimes S \xrightarrow{\eta \otimes id} S \otimes S$;

(iii) comultiplication - Δ.

There are three natural Hopf projections $S \otimes S \to S$:

(i) left - $p_\ell : S \otimes S \xrightarrow{id \otimes \varepsilon} S \otimes k = S$;

(ii) right - $p_r : S \otimes S \xrightarrow{\varepsilon \otimes id} k \otimes S = S$;

(iii) multiplication - m.

4.4. Using the theorem it is easy to check which pairs of structures are compatible. The results are shown in the following table, where plus marks the compatibility:

	i_ℓ	i_r	Δ
p_ℓ	$+$	$-$	$+$
p_r	$-$	$+$	$+$
m	$+$	$+$	$-$

4.5. Let V_1, V_2 be two S-tetramodules. The space $V_1 \otimes V_2$ has the natural structure of an $(S \otimes S)$-tetramodule. Using the discussion above, we can introduce various S-tetramodule structures on $V_1 \otimes V_2$. Our goal is to give a natural definition of tensor product in the category of tetramodules. From this point of view the space $V_1 \otimes V_2$ is "too big". Its $(S \otimes S)$-tetramodule structure is "S times too much" for an S-tetramodule. The definition of the tensor product is given in the next section.

5. Tensor products of S-tetramodules.

5.1. Let $V_1,\ V_2$ be two S-tetramodules. Denote $W = V_1 \otimes_S V_2$. We introduce an S-bimodule structure on W by following formulas:

$$\begin{aligned} m_\ell : \qquad & S \otimes W \to W \\ & (f, v_1 \otimes v_2) \mapsto (fv_1 \otimes v_2) \end{aligned}$$

$$\begin{aligned} m_r : \qquad & W \otimes S \to W \\ & (v_1 \otimes v_2, f) \mapsto (v_1 \otimes v_2 f) \end{aligned}$$

and an S-bicomodule structure by:

$$\begin{aligned} s_\ell(h)(v_1 \otimes v_2) &= s_\ell(h)v_1 \otimes s_\ell(h)v_2 \\ s_r(h)(v_1 \otimes v_2) &= s_r(h)v_1 \otimes s_r(h)v_2 \ , \end{aligned}$$

where $s_l(h)$ ($s_r(h)$) is the left (right) action of the point h of the torus H.

Statement. These S-bimodule and S-bicomodule structures are correctly defined and compatible.

Therefore, $W = V_1 \otimes_S V_2$ is equipped with the natural S-tetramodule structure.

5.2. Let V_1^H and V_2^H be the spaces of right H-invariants in V_1 and V_2. Then $W = V_1 \otimes_S V_2$ is isomorphic to $V_1^H \otimes V_2^H \otimes S$. This isomorphism could be realized through the map:

$$V_1^H \otimes V_2^H \otimes S \to (V_1^H \otimes 1\!\!1) \otimes (V_2^H \otimes S) \subset V_1 \otimes V_2 \to V_1 \otimes_S V_2 \ .$$

The S-tetramodule structure on W can be described as follows: $W^H \approx V_1^H \otimes V_2^H$ is the space of right H-invariants in W. The adjoint action of H on W^H equals

$$ad_H \mid_{V_1} \otimes ad_H \mid_{V_2}$$

and the operator $L(\lambda)$ on W^H equals

$$L(\lambda) \mid_{V_1} \otimes L(\lambda) \mid_{V_2} \ .$$

5.3. Thus the category of S-tetramodules is a monoidal category and is equivalent to the monoidal category of linear spaces equipped with an algebraic action of H and a commuting action of Λ.

5.4. Let us introduce another tensor product. Let V_1, V_2 be two S-tetramodules. Denote by $V_1 \otimes^S V_2$ a subspace in $V_1 \otimes V_2$ of vectors (v_1, v_2) such that, for any $h \in H$:

$$s_r(h)v_1 \otimes s_\ell^{-1}(h)v_2 = v_1 \otimes v_2 \ .$$

We introduce an S-bimodule structure on $W = V_1 \otimes^S V_2$ by the following formulas:

$$\begin{aligned} m_\ell : \qquad & S \otimes W \to W \\ & (f, v_1 \otimes v_2) \mapsto \Delta f \cdot (v_1 \otimes v_2) \end{aligned}$$

$$\begin{aligned} m_r : \qquad & W \otimes S \to W \\ & (v_1 \otimes v_2, f) \mapsto (v_1 \otimes v_2)\Delta f \end{aligned}$$

and an S-bicomodule structure by:

$$\begin{aligned} s_\ell(h)(v_1 \otimes v_2) &= s_\ell(h)v_1 \otimes v_2 \\ s_r(h)(v_1 \otimes v_2) &= v_1 \otimes s_r(h)v_2 \ . \end{aligned}$$

Statement. These S-bimodule and S-bicomodule structures are correctly defined and compatible.

So W is equipped with the natural S-tetramodule structure.

5.5. *Lemma.* $V_1 \otimes_S V_2$ and $V_1 \otimes^S V_2$ are canonically isomorphic.

Proof. Let V_1^H be the space of right H-invariants in V_1 and ${}^H V_2$ be the space of left H-invariants in V_2. Then $V_1 = V_1^H \otimes S$ and $V_2 = S \otimes {}^H V_2$. A natural projection $V_1 \otimes V_2 \to V_1 \otimes_S V_2 = V_1^H \otimes S \otimes {}^H V_2$ is given by

$$V_1^H \otimes (S \otimes S) \otimes {}^H V_2 \stackrel{\mathrm{id} \otimes m \otimes \mathrm{id}}{\longrightarrow} V_1^H \otimes S \otimes {}^H V_2.$$

We can describe the induced S-tetramodule structure on $V_1^H \otimes S \otimes {}^H V_2$ as follows. $V_1^H \otimes S \otimes {}^H V_2 = V_1 \otimes {}^H V_2$, hence the left S-module and S-comodule structures on V_1 define left structures on $V_1^H \otimes S \otimes {}^H V_2$. Symmetrically, $V_1^H \otimes S \otimes {}^H V_2 = V_1^H \otimes V_2$, hence the right structures on V_2 define right structures on $V_1^H \otimes S \otimes {}^H V_2$.

We have the natural imbedding $V_1^H \otimes S \otimes {}^H V_2 \to V_1 \otimes V_2$

$$V_1^H \otimes S \otimes {}^H V_2 \overset{\mathrm{id} \otimes \Delta \otimes \mathrm{id}}{\longrightarrow} (V_1^H \otimes S) \otimes (S \otimes {}^H V_2) \ .$$

It is easy to check that this imbedding gives us an isomorphism of S-teramodules $V_1^H \otimes S \otimes {}^H V_2$ and $V_1 \otimes^S V_2$. So $V_1 \otimes_S V_2$ and $V_1 \otimes^S V_2$ are both canonically isomorphic to $V_1^H \otimes S \otimes {}^H V_2$.

5.6. *Remark.* $V_1 \otimes_S V_2$ and $V_1 \otimes^S V_2$ are canonically isomorphic, but their definitions seem to be different. These definitions are dual in some sense which we will not discuss here.

6. Universal graded Hopf algebra.

6.1. Let us return to a Hopf algebra A with a Hopf ideal I, such that $gr_0 A$ equals S. We denote $gr_1 A$ by T. Algebra grA is generated by $S \oplus T$.

6.2. *Lemma.* Given an S-tetramodule T, there exist a graded Hopf algebra $\tilde{A}(S, T)$, such that $\tilde{A}_0 = S$, $\tilde{A}_1 = T$, $\tilde{A}$ supplies T with the given S-tetramodule structure; and $\tilde{A}$ is universal with respect to these properties. The Hopf algebra $\tilde{A}$ is defined up to a canonical isomorphism.

6.3. Explicitly, the universal Hopf algebra $\tilde{A}(S, T) = \oplus \tilde{A}_n$ can be described as follows: $\tilde{A}_n$ equals $T \otimes_S T \otimes_S \ldots \otimes_S T$ (n factors) and has an S-tetramodule structure described in 5. The multiplication is natural:

$$(\tilde{A}_n, \tilde{A}_m) \to \tilde{A}_n \otimes_S \tilde{A}_m = \tilde{A}_{n+m} \ .$$

6.4. For describing the comultiplication we use the fact that $\tilde{A} \otimes \tilde{A}$ is the graded algebra

$$(\tilde{A} \otimes \tilde{A})_n = \bigoplus_{i=0}^{n} (\tilde{A}_i \otimes \tilde{A}_{n-i}) \ ;$$

and we already have the comultiplication formula for $\tilde{A}_0$ and $\tilde{A}_1$:

$$\begin{aligned} \Delta &: \tilde{A}_0 \to (\tilde{A} \otimes \tilde{A})_0 \\ \Delta &: S \to S \otimes S \\ \Delta s &= s \otimes s \end{aligned}$$

$$\begin{aligned} \Delta &: \tilde{A}_1 \to (\tilde{A} \otimes \tilde{A})_1 \\ \Delta &: T \to S \otimes T + T \otimes S \\ \Delta t &= c_\ell(t) + c_r(t) \ . \end{aligned}$$

The Hopf algebra $\tilde{A}$ is generated by $\tilde{A}_0 \oplus \tilde{A}_1$, so using the fact that the comultiplication is a morphism of algebras, we can easily calculate the comultiplication formula for any element of A: $\Delta(t_1 \otimes t_2)$ is equal to $\Delta t_1 \cdot \Delta t_2$ and so on.

It is easy to prove that this definition is correct and supplies the algebra $\tilde{A}$ with the bialgebra structure.

6.5. *Antipode.* There is an antipode on S. Using the following commuting diagram

$$\begin{array}{ccc} S\otimes T + T\otimes S & \stackrel{i\otimes id}{\longrightarrow} & S\otimes T + T\otimes S \\ \uparrow \Delta & & \downarrow m \\ T & \stackrel{\eta\varepsilon=0}{\longrightarrow} & T \end{array}$$

we can easily define an antipode on T and by induction an antipode on $\tilde{A}$. Thus the bialgebra $\tilde{A}$ is supplied with the Hopf algebra structure.

6.6. *Remark.* The space $\tilde{A} \otimes^S \tilde{A}$ is a subspace in $\tilde{A} \otimes \tilde{A}$. It is easy to check that $\operatorname{Im} \Delta \in \tilde{A} \otimes^S \tilde{A}$.

6.7. If A is a Hopf algebra which corresponds to the same S-tetramodule T, then we have a natural morphism of Hopf algebras:

$$\tilde{A}(S,T) \to gr\, A\ .$$

7. Examples.

7.1. Below we list some examples of a Hopf algebra A with a natural Hopf ideal I, $A/I = S$. We describe an S-tetramodule $T = I/I^2$.

7.2. *Lie case.* Let G be a reductive algebraic group, H its Cartan subgroup. Let $A = \mathbb{C}[G]$ be the Hopf algebra of regular functions on G and I an ideal of functions equal to 0 on H. Then $S = A/I$ equals $\mathbb{C}[H]$, $T = I/I^2$ is an S-tetramodule. The space T^H is isomorphic to $(g/h)^*$.

As an H-module, T^H is a direct sum of one-dimensional representations T^H_α which correspond to roots of G. An S-bimodule structure of V is trivial, which is equivalent to the fact that the lattice λ acts on T^H as the identity.

7.3. $SL_q(n)$. The algebra A of functions on $SL_q(n)$ is defined as an algebra generated by n^2 noncommuting elements a_{ij} $(1 \le i,j \le n)$, satisfying the following relations [M]:

Introduce matrices

$$Y(i,j,k,l) = \begin{pmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix}.$$

Then for every $1 \leq i < k \leq n$, $1 \leq j < l \leq n$ we put relations:

$$\begin{matrix} YQY^t = x_1 Q \\ Y^t QY = x_2 Q \end{matrix} \qquad x_1, x_2 \in \mathbb{C}^* \,.$$

We have one more relation for the determinant:

$$\sum_{s \in S_n} (-q)^{-l(s)} a_{1\ s(1)} \cdots a_{n\ s(n)} = 1.$$

To define the comultiplication in A we consider the $n \times n$ matrix $Z = \{a_{ij}\}$. Using the natural imbeddings $i',\ i'' : A \to A \otimes A,\ \big(i'(x) = x \otimes 1,\ i''(x) = 1 \otimes x\big)$, we can write the comultiplicaton formulas as:

$$\Delta(Z) = i'(Z) \cdot i''(Z) \ ,$$

which is an equality in $\mathrm{Mat}(n, A \otimes A)$.

The ideal I is generated by $a_{ij} \quad (i \neq j)$. Then A/I is isomorphic to $\mathbb{C}[H]$, where H is an $(n-1)$-dimensional torus. We define T equal to I/I^2. It is easy to see that T^H is a direct sum of one-dimensional representations T^H_α, where α or $-\alpha$ is a simple root.

It is easy to check that a character $\lambda \in \Lambda$ acts on T_α multiplyingit by $q^{-\alpha(\lambda)}$ for $\alpha > 0$ and $q^{\alpha(\lambda)}$ for $\alpha < 0$.

7.4. *Quantum groups of $SL(2)$-type.* In the paper [B-KH1] we constructed quantum groups of $SL(2)$-type. Namely, these groups where attached to S – the space of regular functions on the one-dimensional torus and an S-tetramodule T. The space T^H of H-invariants is two-dimensional: $T^H = T^H_\alpha \oplus T^H_{-\alpha}$ (weight of α is equal to n) and a character s (the basis in S) acts on T^H_α multiplying it by q_α, where $q^n_\alpha = q^n_{-\alpha}$.

The corresponding Hopf algebra is generated by elements $\hat{h},\ h \in H \approx \mathbb{C}^*$ and by elements E and F satisfying the relations:

$$\begin{aligned} \hat{h}_1 \hat{h}_2 &= \widehat{h_1 h_2} \\ \hat{h} E &= \alpha(h) E \hat{h} = h^n E \hat{h} \\ \hat{h} F &= -\alpha(h) F \hat{h} = h^{-n} F \hat{h} \\ [E, F] &= \tfrac{\hat{q}_\alpha - \hat{q}^{-1}_{-\alpha}}{q_\alpha - q^{-1}_{-\alpha}} \\ \Delta \hat{h} &= \hat{h} \otimes \hat{h} \\ \Delta E &= \hat{q}_\alpha \otimes E + E \otimes 1 \\ \Delta F &= 1 \otimes F + F \otimes \hat{q}^{-1}_{-\alpha}. \end{aligned}$$

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