Polynomials

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Class Discussion

Solve the equation $x^3 + x - 2 = 0$. Find the remainder, when $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ is divided by $(x - 1)$. If $P(x)$ is a polynomial, then $P(x) = (x-a)Q(x)+P(a)$, where $Q(x)$ is a polynomial with a smaller degree. It follows that if a is a root, then $P(x) = (x-a)Q(x)$. If a monic polynomial with integer coefficients has a rational root, then the root is an integer that divides the constant term.

Warm-Up

These three problems are from the book *The Riddle of Scheherazade* by Raymond Smullyan. In each of them a robbery happened and there were three suspects: Abu, Ibn and Hasib. Exactly one suspect was guilty.

Exercise 1. Abu: "I didn't commit the robbery!" Ibn: "Hasib certainly didn't!" Hasib: "Yes, I did!" Later on two of them confessed to having lied. Who committed the robbery?

Exercise 2. Ibn: "Hasib never committed the robbery." Hasib: "That is true." Abu: "Ibn is innocent." Curiously enough, the actual thief told the truth, but they didn't all tell the truth. Which one committed the robbery?

Exercise 3. Abu claimed to be innocent; Ibn agreed that Abu was innocent; and Hasib claimed that he himself was the guilty one. As it turned out, the guilty one lied. Which one was guilty?

Polynomials

Exercise 4. AHSME. Suppose $(3x-1)^7 = a_7x^7 + a_6x^6 + \cdots + a_1x + a_0$ for constants $a_0,\ a_1, \ldots, a_7$. Find $a_7 + a_6 + \cdots + a_1 + a_0$.

Exercise 5. ARML. $f(x)$ is a polynomial of degree greater than 3. If $f(1) = 2,\ f(2) = 3$ and $f(3) = 5$, find the remainder when $f(x)$ is divided by $(x-1)(x-2)(x-3)$.

Exercise 6. For what a the polynomial $P(x) = x^{1000} + ax^2 + 9$ is divisible by $x + 1$?

Exercise 7. Find a and b so that the polynomial $P(x) = (a+b)x^5 + abx^2 + 1$ is divisible by $x^2 - 3x + 2$.

Exercise 8. Find the roots of $f(x) = x^3 - 5x^2 - 77x + 441$.

Exercise 9. Simplify $t \cdot \frac{1+\frac{2}{\sqrt{t+4}}}{2-\sqrt{t+4}} + \sqrt{t+4} + \frac{4}{\sqrt{t+4}}$.

Exercise 10. Solve the equation: $\frac{x^2+x-5}{x} + \frac{3x}{x^2+x-5} + 4 = 0$.

Challenge Problems

Exercise 11. Solve the equation: $(x+3)^4 + (x+5)^4 = 16$.

Exercise 12. Solve the equations $x^4 - x^3 - 22x^2 + 16x + 96 = 0$ and $x^3 - 2x^2 - 3x + 10 = 0$, using the fact that they share a root.

Exercise 13. The sultan decides to test his hundred wizards. Tomorrow at noon he will randomly put a red or a blue hat — for both of which he has an inexhaustible supply — on every wizard's head. Each wizard will be able to see every hat but his own. The wizards will not be allowed to exchange any kind of information whatsoever. At the sultan's signal, each wizard needs to write down the color of his own hat. Every wizard who guesses wrong will be executed.

The wizards are all very good friends with each other. They decide that executions are very sad events and they do not wish to witness their friends' deaths. They would rather die themselves. They realize that they will only be happy if all of them survive together. Suggest a strategy that maximizes the probability of them being happy, that is, the probability that all of them will survive.