Sums of Squares

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Class Discussion

Squares have remainders 0 and 1 mod 4 or mod 3. A sum of two squares can't have a remainder 3 mod 4. Fermat's theorem: an odd prime p can be a sum of two squares if and only if p is congruent to 1 mod 4.

Brahmagupta-Fibonacci identity: $(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2$.

Warm-Up

Exercise 1. An astrologist believes that a year is happy if its digit representation contains four consecutive digits. For example, the next year, 2013, will be happy. When was the previous happy year?

Exercise 2. When a four-digit number is multiplied by 4, the result is consists of the same digits in reverse order. Find the number.

Exercise 3. Integers from 1 to 60 are concatenated into one string: $1234\ldots5960$. Erase one hundred digits, so that when you squeeze the leftover digits together, you get the largest number possible. What is this number?

Exercise 4. A five-digit integer *abcde* consist of all distinct integers that sum up to 10. The sum of *abcde* and its reverse, *edcba*, is divisible by 11111. Find *abcde*, if it is known that it is divisible by 7.

Competition Practice

Exercise 5. Integers from 1 to 2012 inclusive are written on the board. In one step you are allowed to erase any two of the numbers and replace them with their difference. After many steps one number is left on the board. Is it even or odd?

Exercise 6. HMNT 2008. Guts Round. What is the largest x such that x^2 divides $24 \cdot 35 \cdot 46 \cdot 57$?

Exercise 7. HMNT 2008. Guts Round. You have a die with faces labelled 1 through 6. On each face, you draw an arrow to an adjacent face, such that if you start on a face and follow the arrows, after 6 steps you will have passed through every face once and will be back on your starting face. How many ways are there to draw the arrows so that this is true?

Exercise 8. HMNT 2008. Guts Round. John M. is sitting at $(0, 0)$, looking across the aisle at his friends sitting at (i, j) for each $1 \leq i \leq 10$ and $0 \leq j \leq 5$. Unfortunately, John can only see a friend if the line connecting them doesn't pass through any other friend. How many friends can John see?

Exercise 9. HMNT 2008. Individual Round. How many diagonals does a regular undecagon (11-sided polygon) have?

Challenge Problems

Exercise 10. There is a ceiling a hundred feet above you that extends forever, and hanging from it side-by-side are two golden ropes, each a hundred feet long. You have a knife, and would like to steal as much of the golden ropes as you can. You are able to climb ropes, but not survive falls. How much golden rope can you get away with, and how? Assume you have as many hands as you like.