# Number of Divisors

Tanya Khovanova

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## Class Discussion

An integer has an odd number of divisors iff it is a square. An integer is *square-free* (not divisible by any square) iff its number of divisors is a power of two.

Denote $\tau(n)$ the number of factors of $n$. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is a prime decomposition of $n$, then the number of divisors of $n$ is:

$$\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

The product of all factors of $n$ is:

$$n^{\tau(n)/2}.$$

## Warm-Up

**Exercise 1.** No one is standing in the room, but rather every person is sitting on a three-legged stool or a four-legged chair. There are 39 legs total in the room and no places to seat are left. How many stools are there?

**Exercise 2.** Do there exist natural numbers $x$, $y$, and $z$ satisfying the equation: $28x + 30y + 31z = 365$?

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**Exercise 3. HMNT 2011, Guts round 7 points.** How many ordered triples of positive integers $(a, b, c)$ are there for which $a^4 b^2 c = 54000$?

**Exercise 4. AIME 1988.** Compute the probability that a randomly chosen positive divisor of $10^{99}$ is an integer multiple of $10^{88}$.

**Exercise 5.** Determine the product of distinct positive divisors of 120. What about 100?

**Exercise 6.** Determine the number of ordered pairs of positive integers $(a, b)$ such that the least common multiple of $a$ and $b$ is $2^3 5^7 11^{13}$.

**Exercise 7. HMNT 2011, Guts round 10 points.** For a positive integer $n$, let $p(n)$ denote the product of the positive integer factors of $n$. Determine the number of factors $n$ of 2310 for which $p(n)$ is a perfect square.

**Exercise 8.** Find all natural numbers that are divisible by 30 and have exactly 30 distinct divisors.

## Competition Practice

**Exercise 9. HMMT 2011, Guts round. 8 points.** Rosencrantz and Guildenstern play a game in which they repeatedly flip a fair coin. Let $a_1 = 4$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. On the $n$th flip, if the coin is heads, Rosencrantz pays Guildenstern $a_n$ dollars, and, if the coin is tails, Guildenstern pays Rosencrantz $a_n$ dollars. If play continues for 2010 turns, what is the probability that Rosencrantz ends up with more money than he started with?

**Exercise 10. HMNT 2011, Guts round 8 points.** For positive integers $m$, $n$, let $\gcd(m, n)$ denote the largest positive integer that is a factor of both $m$ and $n$. Compute $\sum_{n=1}^{91} \gcd(n, 91)$.