Combinatorial Identities

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Two math teachers had a fight. It seems they couldn't divide something.

Class Discussion

Combinatorial Identities. Pascal's triangle. Combinatorial and formulae proofs:

- Symmetry: $\binom{n}{k} = \binom{n}{n-k}$ for $0 \le k \le n$.
- Pascal's rule: $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.
- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
- Two teams with m and k people: $\binom{n}{m}\binom{n-m}{k} = \binom{n}{k}\binom{n-k}{m} = \binom{n}{k+m}\binom{k+m}{m}$.

Warm-Up

Exercise 1. I'm a four-digit number! My 2nd digit is twice greater than my 3rd. The sum of all my digits is thrice greater than my last digit! The product of my 3rd and 4th digits is 12 times greater than the ratio of my 2nd to 3rd. What am I?

Exercise 2. Three men were in a boat. It capsized but only two got their hair wet. Why?

Combinatorial Identities

Exercise 3. Prove all the identities above using formulae.

Exercise 4. Find a combinatorial way to prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

Exercise 5. Why do powers of 11 look like lines in the Pascal's triangle: $11^2 = 121$ and $11^3 = 1331$? How much is 11^4? Will the pattern continue? Explain.

Exercise 6. Suggest a way to continue the Pascal's triangle up.

Competition Practice

Exercise 7. MAML 2006. Draw six cards from a standard deck of 52 playing cards without replacement. How many distinct ways can you choose these six cards so that:

- The first card drawn is a spade,
- the second card drawn is also a spade,
- the third card drawn is a club,
- the fourth card drawn is a diamond,
- the fifth card drawn is a red card (either a diamond or a heart), and
- the sixth and final card drawn is an ace?

Exercise 8. MAML 2003. A sequence u is defined recursively as follows:

- $u_0 = 4$
- $u_1 = 7$
- $u_{n+2} = 5u_{n+1} - 6u_n$ for all $n \geq 2$.

Find the value of u_x for all $x \geq 2$. Your answer should be a function of x.

Exercise 9. My friend's phone number has area code 617. Unfortunately, in my phone book I have only six digits for the rest of the number. Somehow I skipped one of the digits. How many possibilities are there for her number?

Challenge Problems

Exercise 10. How many ways are there to place the following chess pieces on a chessboard, so they do not attack each other:

- two rooks
- two bishops
- two knights
- two queens
- two kings