Catalan Numbers

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Class Discussion

Sequences. Online Encyclopedia of Integer sequences: `http://oeis.org/`. Continue the sequence 1,2,3,4,5,6,7,8,9.

- 1 — the digital sums of natural numbers;
- 11 — palindromes;
- 0 — digital products of natural numbers;
- 13 — numbers such that 2 to their powers doesn't contain 0;
- 153 — numbers that are sums of fixed powers of their digits;
- 22 — numbers for which the sum of digits equals the product of digits.

The most versitile sequence. Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. Many definitions:

- Ballot Problem: Suppose A and B are candidates for office and there are $2n$ voters, n voting for A and n for B. In how many ways can the ballots be counted so that A is always ahead of or tied with B?
- Dyck Words: A Dyck word is a string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's.
- Matching Parenthesis: counts the number of expressions containing n pairs of parentheses which are correctly matched.
- Binary Bracketing: Number of ways to insert n pairs of parentheses in a word of $n + 1$ letters.
- Rooted Ordered Binary Trees.
- Monotonic Paths.

Warm-Up

Exercise 1. There are 8 apples on the table, you take 3. How many do you have?

Exercise 2. A box has nine ears of corn in it. A squirrel carries out three ears a day, and yet it takes him nine days to carry out all the corn. Explain?

Catalan Numbers

Exercise 3. Prove that $C_n = \binom{2n}{n} - \binom{2n}{n-1}$.

Exercise 4. Calculate the first five Catalan numbers.

Exercise 5. For which n, Catalan numbers C_n are odd?

Exercise 6. Prove that "Matching Parenthesis" and "Binary Bracketing" define the same sequence.

Competition Practice

Exercise 7. HMMT 2003. We wish to color the integers 1, 2, 3, . . . , 10 in red, green, and blue, so that no two numbers a and b, with $a-b$ odd, have the same color. (We do not require that all three colors be used.) In how many ways can this be done?

Exercise 8. HMMT 2004. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

Challenge Problems

Exercise 9. In the line to the movie theater there are $m + k$ people. Out of these people, m people have only \$5 bills and k people have only \$10 bills. The ticket costs \$5, and at the beginning, the cashier doesn't have any change. Also, the cashier only accepts cash. In how many different ways can these people form a line so that there is no problem with change.