Fermat's Little Theorem

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Nothing produces such odd results as trying to get even.

Class Discussion

Fermat's Little Theorem.

Warm Up

Exercise 1. Mrs. Fullhouse has 2 sons, 3 daughters, 2 cats and 1 dog. How many children does she have?

Exercise 2. 2001 AMC 8. Two dice are thrown. What is the probability that the product of the two numbers is a multiple of 5?

Exercise 3. 2003 AMC 8. How many integers between 1000 and 2000 have all three of the numbers 15, 20 and 25 as factors?

Exercise 4. 2002 AMC 8. What is the measure of the acute angle formed by the hands of a clock at 4:20 a.m.?

Fermat's Little Theorem

Exercise 5. Find the remainder of 3^{63}, when divided by 61.

Exercise 6. Prove that for any integer a, $a^5 - a$ is divisible by 30 and $a^{11} - a$ is divisible by 66.

Exercise 7. Given a sequence $a_n = 1 + 2^n + 3^n + 4^n + 5^n$. Is it possible to find five consecutive terms of this sequence such that all of them are divisible by 2010?

Exercise 8. Prove that for any integer n that is not a multiple of 17, that either n^8+1 or n^8-1 is divisible by 17.

Exercise 9. The sum of three integers a, b and c is divisible by 30. Prove that $a^5+b^5+c^5$ is divisible by 30.

Exercise 10. Let p be a prime number and a is not a multiple of p. Prove that there exists an integer b such that $ab \equiv 1 \pmod{p}$.

Competition Practice

Exercise 11. 2002 HMMT. Find the greatest common divisor of the numbers $2002+2$, 2002^2+2, 2002^3+2, $\ldots$.

Exercise 12. 1999 HMMT. A combination lock has a 3 number combination, with each number an integer between 0 and 39 inclusive. Call the numbers n_1, n_2, and n_3. If you know that n_1 and n_3 leave the same remainder when divided by 4, and n_2 and n_1+2 leave the same remainder when divided by 4, how many possible combinations are there?

Exercise 13. 2007 HMMT. Define the sequence of positive integers a_n recursively by $a_1 = 7$ and $a_n = 7^{a_{n-1}}$ for all $n \geq 2$. Determine the last two digits of a_{2007}.

Exercise 14. 2002 HMMT. For how many integers a ($1 \leq a \leq 200$) is the number a^a a square?

Challenge Problems

Exercise 15. A king decides to give 100 of his prisoners a test. If they pass, they can go free. Otherwise, the king will execute all of them. The test goes as follows: The prisoners stand in a line, all facing forward. The king puts either a black or a white hat on each prisoner. The prisoners can only see the colors of the hats in front of them. Then, in any order they want, each one guesses the color of the hat on their head. Other than that, the prisoners cannot speak. To pass, no more than one of them may guess incorrectly. If they can agree on their strategy beforehand, how can they be assured that they will survive?