How to Cut a Cake

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Class Discussion

Dividing treasure between three people.

Warm-Up

Exercise 1. You are running a race and you passed the person who was running second. What place are you now?

Exercise 2. You are running a race and you passed the person who was running last. What place are you now?

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Exercise 3. A fair division procedure is called *envy-free* if after the division no one wants to swap his/her part for someone elses. For three people it means each person believes s/he got at least one third and no one got more than s/he. For each strategies below decide whether it is envy-free.

- Alice divides the cake in two equal parts. Bob chooses the part he likes. Alice divides her part into three equal pieces. Bob divides his part into three equal pieces. Carol chooses one piece from Alice and one piece from Bob.
- Alice cuts one third from the cake. If Bob think that the part Alice cut is bigger than one third, he cuts a smaller part to make it one third. Then Carol looks at the result and if she thinks that the part is more than one third, she cuts one third. The last person to cut, takes the piece. The other two people divide what is left buy the divide-and-choose algorithm.

Competition Practice

There is a relay round at ARML. Each Relay team member will receive a different problem. TNYWR stands for "the number you will receive". When the first person solves his (or her) problem, s/he writes the answer on a piece of paper and passes it to the second person and so on. The second person needs that number to solve his/her problem (the number is referred to as "the number you will receive," or TNYWR). The problems below are from ARML 2010 Super Relay in the relay order.

Exercise 4. Let N be a perfect square between 100 and 400, inclusive. What is the only digit that cannot appear in N?

Exercise 5. Let $T = TNYWR$. Let A and B be distinct digits in base T, and let N be the largest number of the form ABA_T. Compute the value of N in base 10.

Exercise 6. Let $T = TNYWR$. Given a nonzero integer n, let $f(n)$ denote the sum of all numbers of the form i^d, where $i = \sqrt{-1}$, and d is a divisor (positive or negative) of n. Compute $f(2T + 1)$.

Exercise 7. Let $T = TNYWR$. Compute the real value of x for which there exists a solution to the system of equations

$$\begin{aligned} x + y &= 0 \\ x^3 - y^3 &= 54 + T. \end{aligned}$$

Exercise 8. Let $T = TNYWR$. In $\triangle ABC$, $AC = T^2$, $\angle ABC = 45°$, and $\sin \angle ACB = 8/9$. Compute AB.

Exercise 9. Let $T = TNYWR$. Given two tangent circles, the smaller circle is internally tangent to the larger circle at point O, and $\overline{OP}$ is a diameter of the larger circle. Point Q lies on $\overline{OP}$ such that $PQ = T$, and $\overline{PQ}$ does not intersect the smaller circle. If the larger circle's radius is three times the smaller circle's radius, find the least possible integral radius of the larger circle.

Exercise 10. Let $T = TNYWR$. The sequence $a_1, a_2, a_3, \ldots$ is an arithmetic progression, d is the common difference, $a_T = 10$, and $a_K = 2010$, where $K > T$. If d is an integer, compute the value of K such that $|K - d|$ is minimal.