Remainders Modulo 9

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Class Discussion

Romeo—Romeo.

- The sum of digits of n has the same remainder modulo 9 as n.
- Ten-digit numbers with all distinct digits are divisible by 9.
- Rearranging digits doesn't change the remainder modulo 9.
- Removing the leading digit and adding it to the rest of the the number doesn't change the remainder mod 9.

Warm Up

Exercise 1. Let $SOD(n)$ be the sum of the digits of n. Suppose $f(n)$ is the result of iterating SOD many times until we get a single digit. That is, $f(n) = SOD(SOD(SOD \dots (n) \dots))$.

- Find $f(2^{2010})$.
- Find $f(3^{2010})$.
- Find $f(4^{2010})$.
- Find $f(5^{2010})$.
- Find $f(6^{2010})$.
- Find $f(7^{2010})$.
- Find $f(8^{2010})$.
- Find $f(9^{2010})$.
- Find $f(1234^{2009})$.

Exercise 2. A frog jumps along the line. First it jumped 1 cm, then 3 cm in the same or the opposite direction, then 5 cm. It continues with the sequence of odd numbers. Can it be back at the beginning after 14 jumps?

Exercise 3. Prove that for any natural number n, $4^n + 15n - 1$ is divisible by 9.

Exercise 4. Proof that the sum of the digits of a square can't be 1967.

Exercise 5. A number has three ones. All other digits are zeroes. Can it be a square?

Review

Exercise 6. In the year X a certain day of the mongth was never a Sunday. What day was that?

Exercise 7. Start with 7^{2010}. At each step, delete the leading digit, and add it to the remaining number.

- Repeat until a number with exactly 10 digits remains. Prove that this number has two equal digits.
- Repeat until you get a single digit. What is it?

Exercise 8. Is it possible for two different powers of 2 to have the same digits (in a different order)?

Competition Practice

Exercise 9. 1967 USSR Olympiad. Number b was produced by permuting digits in number a. Can $a + b$ equal $999\ldots999$, a number written with 1967 nines? In a similar setting, prove that if $a + b = 10^{10}$ then a is divisible by 10.

Exercise 10. 1962 IMO. Determine the smallest possible integer x whose last decimal digit is 6, and if we erase this last 6 and put it in front of the remaining digits, we get four times x.