**Kids at the Movies**

The probability that an ordered pair of kids is mixed is (8x7 +7\*8)/15x14 = 8/15; thus each of the 14 adjacent pairs of seats has 8/15 mixed pairs in it on average. Since expectations add regardless of dependence, the expected number of mixed adjacent pairs is 14x8/15 = 212/15.

**100 passengers on a train**

Let us denote by Fred the passenger who, after everyone is randomly seated, suddenly decides to demand his reserved seat. It suffices to compute the probability *p* that some other passenger, say Clara, gets bumped; then the expected number *b* of passengers that get bumped is 99*p*.

To get a uniformly random seating, the order in which the passengers arrive is irrelevant as long as each passenger takes a uniformly random unoccupied seat. Suppose Clara gets on first, picks a random seat, and looks at the reservation tag on the seat; she then calls out to the person whose name she finds, say Donald. Donald gets on next and picks a random unoccupied seat, again calling out the name of the seat’s legal owner, who gets on next. This continues until Clara’s own name is called, closing the cycle. The rest of the passengers then board in any order.

Going back to Clara’s cycle, let’s interrupt the procedure when either her name or Fred’s is called. Since the called names are random among those not yet called, it’s a tossup which event occurs first. But if Fred’s name is called first, Clara will later be bumped: Fred will bump the person who called his name, who will bump the person who called him, etc. until Clara is uprooted. On the other hand, if Clara’s name is called first, the cycle is closed and Fred’s cycle can’t touch Clara.

Thus Clara’s probability of being bumped is ½ and the expected number of bumped passengers is 99(1/2) = 49.5.

(Using the fact that in a random permutation the size of the cycle containing a fixed element is uniformly random, which can be proved several ways, you can deduce the stronger fact that the number of passengers who get bumped is uniformly random from 0 to 99.)

**Loaded dice**

Suppose these dice exist and let *p*1, . . . , *p*6 be the probabilities of rolling 1, . . . , 6 respectively with the first die, and *q*1, . . . , *q*6 with the second. Then *p*1*q*1 = *p*6*q*6 = 1/11 since these represent the only way to roll a sum of 2 or 12. If *p*1 > *p*6 then *q*1 < *q*6 and *p*1*q*6 > 1/11; if *p*1 < *p*6 then *q*1 > *q*6 and *p*6*q*1 > 1/11; if *p*1 = *p*6 then *q*1 = *q*6 and *p*6*q*1 = *p*1*q*6 = 1/11. But the probability of rolling a sum of 7 is at least *p*6*q*1 + *p*1*q*6 which exceeds 1/11 in all three cases, a contradiction.

A similar argument shows that you can’t bend two coins so that the probabilities of getting 0, 1 or 2 heads when you flip them are all equal.

(These facts can also be proved using generating functions, nut, again, we are going for the most elementary arguments here.)

**Green and red apples**

When choosing successive items without replacement, it is often convenient to put all the randomness in one place by putting the items in random order and then taking them left to right. For this puzzle, the desired probability becomes simply the probability that the rightmost apple is green. Since we may as well choose the random order from right to left, that probability is just M/(M+N).

**Determinant of a** {+1,-1} **matrix**

Since this is supposed to be a probability problem, it is easy to guess that we should take the matrix M to be uniformly random; then each entry Mi,j will be independently equal to +1 or -1 with probability ½.

If σ is a permutation of {1, . . . , 11}, we will denote by <σ> the product of the entries Mi,σ(i) times the sign of σ; thus the determinant D is the sum of <σ> over all σ. Of course the expected value of <σ> is 0 and its variance is 1; moreover, for distinct σ and τ, <σ> and <τ> are independent since each contains a random factor not in the other.

The variances of pairwise-independent random variables add (as is easily seen by expanding the expectation of the square). It follows that the variance of D is 11!, therefore its standard deviation is the square root of 11! which is more than 4000. Thus there is some matrix for which |D|>4000; if this D is negative we can just flip the signs in any row or column to make it positive.

(To see without a calculator that 11! > 40002, group their factors as follows and compare group by group: (11x10)x(9x8)x(7x6x5x4)x(3x2) versus (10x10)x(8x8)x(5x5x5x5)x(2x2). In fact, the square root of 11! is about 6318.)

**Moving points**

The first thing to notice is that it doesn’t matter if the points pass through each other instead of bouncing, and therefore the length of time it takes for the last point to hit an end is the maximum length that any point sees ahead of it. This is the same as the maximum of *n* uniformly random reals between 0 and 1, whose expectation is *n*/(*n*+1).

To see without calculus that this is so, place n+1 points at random on a unit-circumference circle; the mean distance between consecutive points is then 1/(*n*+1). Open the circle into a unit interval by splitting at one of the points, and we have that the mean distance from the rightmost interior point to the right-hand end is 1/(*n*+1).

**The Infinite Table**

Let C be the event that Vasya knows the answer to begin with, R the event that he gets it from his right-hand neighbor, L the event that he gets it from his left neighbor. In order to make these events independent, we assume that information travels only *toward* Vasya, not away; in other words, that no one to Vasya’s right peeks left, and no one to his left peeks right. This has no effect on Vasya’s chances.

Let *p* be the probability of R (and thus, by symmetry, also of L). R occurs if (1) Vasya peeks right (probability ¼) and (2) either his right-hand neighbor knew the answer herself (probability ½) or she didn’t, but was able to get it from her right-hand neighbor (probability ½ x *p*). This gives p = 1/4 (1/2 + p/2), p = 1/7.

Now we have that the probability of L or C or R equals P(L) + P(C) + P(R) – P(L and C) – P(L and R) – P(C and R) + P(L and C and R) = 1/7 + ½ + 1/7 – 1/14 – 1/49 – 1/14 + 1/99 = 31/49.

 -PW 5/31/14